

# Contemplating visualization as an epistemological learning tool in mathematics

Norma Presmeg

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**Abstract** In this commentary, I use thirteen questions generated previously, concerning significant aspects of visualization in mathematics education that are in need of further research, to analyze the contributions of papers in this issue. After a descriptive section addressing how the papers investigate visualization as an epistemological learning tool, two further sections analyze converging knowledge and missing aspects, respectively. Finally, suggestions are provided for moving the field forward with regard to both theoretical and methodological aspects.

## 1 Introduction

When I accepted the challenge of writing a commentary on the interesting but diverse set of papers in this issue, I knew that the field had moved forward considerably since I started research in this area in the early 1980s, when little research had been carried out on how visualization is implicated in the teaching and learning of mathematics at all levels. However, it is apparent that many of the questions still in need of investigation that I identified in my Handbook chapter (Presmeg 2006a, Nardi 2014) are relevant to the concerns of the authors of papers in this issue, and many of these are still in need of investigation. Thus, in addressing the four questions suggested by the editors (each with its own section in what follows), I shall refer back to this list of thirteen significant questions:

1. What aspects of pedagogy are significant in promoting the strengths and obviating the difficulties of use of visualization in learning mathematics?
2. What aspects of classroom cultures promote the active use of effective visual thinking in mathematics?
3. What aspects of the use of different types of imagery and visualization are effective in mathematical problem solving at various levels?
4. What are the roles of gestures in mathematical visualization?
5. What conversion processes are involved in moving flexibly amongst various mathematical registers, including those of a visual nature, thus combating the phenomenon of compartmentalization?
6. What is the role of metaphors in connecting different registers of mathematical inscriptions, including those of a visual nature?
7. How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions?
8. How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions, and conventional mathematical processes and notations?
9. How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects?
10. How may visualization be harnessed to promote mathematical abstraction and generalization?
11. How may the affect generated by personal imagery be harnessed by teachers to increase the enjoyment of learning and doing mathematics?
12. How do visual aspects of computer technology change the dynamics of the learning of mathematics?

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N. Presmeg (✉)  
Illinois State University, Normal, IL, USA  
e-mail: npresmeg@msn.com

13. What is the structure and what are the components of an overarching theory of visualization for mathematics education?

(Presmeg 2006a, p. 227)

I started to work on “overarching theory” for a presentation on this topic at Topic Study Group 20 of the 11th meeting of the *International Congress on Mathematical Education* (Presmeg 2008a), and the problem of compartmentalization was addressed in research on the learning of trigonometry (Presmeg 2006b), along with further research on metaphors (Presmeg 2008b). But a gap in extant research concerns the role of visualization in the socio-cultural context of mathematics classrooms, and how its power might be promoted in the learning of specific mathematics topics: this lacuna is addressed by many of the papers in this issue, and therein lies their importance. How can teachers promote the use of visual means for effective learning of mathematics in classrooms? It is not enough for teachers to be enthusiastic about the use of visual aids in such learning: the high school teachers in the “visual” group in my research promoted visualization in all forms extensively; but they were insufficiently aware of problems of abstraction and generalization experienced by their pupils with regard to visualization, for such learning to be effective. It was with teachers in the “middle” group (who stressed the generalization aspect) that such learning was most effective. (In classes of the “nonvisual” group of teachers, the visualizers were like fishes out of water, and their learning could not thrive.)

Before I address the content of the individual papers, there are a few general points I would like to raise. The first concerns the title of this issue. I wondered why it was necessary to add the word “epistemological” to *visualization as a learning tool in mathematics*. This word, which can be contrasted with “ontological” in philosophy, seems to imply that a theoretical or philosophical stance is taken in general. If it is intended to signify that the focus is not the *reality* of visualization (ontology), but a *theory of cognition* (epistemology) concerning visualization, then the purpose is clear. Funk and Wagnall (2003) defined epistemology as the department of philosophy that “investigates critically the nature, grounds, limits, and criteria, or validity, of human knowledge; theory of cognition” (p. 428). In any case, it is evident that the role of (at least some aspects of) visualization in the learning of mathematics in classrooms is the focus of each of the papers in this issue. Many of the authors embrace the definition of visualization given by Arcavi (2003), which is broad enough to include product and process, visualization as an artifact (as in the number line as a tool of learning), as well as the meanings constructed by individual learners. I return to the issue of definition of visualization in the final section of this commentary.

Paivio (1971) suggested that the way a task is tackled by an individual depends on the following: the task itself (in this case whether it is presented in visual form or not); instructions to do the task in a certain way; and finally, individual differences. I would add a fourth item to his list: the sociocultural climate of the classroom (whether or not visualization is accepted, encouraged, and valued). All the papers in this issue address, in one way or another, the task itself, and the nature of the instruction; and some of the papers include social or cultural aspects. However, although they take into account different ways of *interpreting* visual displays, none of the papers take into account individual differences in *preference* for visual thinking. According to the results over many decades of giving my “preference for visualization” test (e.g., Presmeg 2006a) to students and their teachers in high schools, and to university students, the frequency of preference for visual thinking scores in the general population follows a normal, Gaussian, distribution. In other words, for most people, the task itself, instructions, and sociocultural factors influence whether they attempt a visual solution when there is the choice. But at the two far ends of the distribution, there are visualizers who *always* try to make an image or picture if it is not given, and also nonvisualizers who do not feel this need at all. In the light of these individual differences in need for visualization amongst mathematics learners—and indeed in the types of visualization that they spontaneously create—caution should be exercised in claims about the general efficacy of visual tools for a class of mathematics learners.

## 2 How is visualization as an epistemological learning tool described in this set of papers?

The first four papers after the introduction all involve various forms of the *number line* as a visual learning tool. Steenpass and Steinbring (2014) use the conceptual framework of “frame based interpreting competence” to conduct a fine-grained analysis of the meanings attributed to a number line with strokes of various lengths (without numerals attached) to mark off—flexibly—units, fives, and tens, or other appropriate denominations. There is also a suggestive arc linking some of these strokes, thus introducing the idea that some operation is being denoted. Using Sonja’s case, they emphasize the theoretical ambiguity of visual aids such as this one (the *polysemy* identified by Abrahamson et al. 2014). Their research reinforces the inadequacy of teachers merely *showing* learners how to use such a diagram in a flexible way. The value of social interaction in the classroom for modifying and expanding the frames that children use in interpreting such a task is pointed out.

Without using the theoretical construct of frames, the paper by Gellert and Steinbring (2014) reinforces the efficacy of small-group interaction in primary school classrooms, in providing opportunities for negotiating and clarifying differences in the variety of ways that learners interpret number lines with strokes and “boxes” as signs for particular values. Using elements of Steinbring’s epistemological triangle, they illustrate the reflexive relationships between the object/reference context, and the sign/symbol, and the difficulties that learners experience in interpreting these relationships. I gained the impression that *visualization* as such is less the focus of this paper than is the value of small-group discussions. However, the results reported resonate well with the research reported in the next paper, by Margolinas and Wozniak (2014), who worked with younger, pre-school children, in the context of threading beads on a string with a knot as starting point. These young children could recognize number as quantity, but many had difficulty with the concept of *position*. Despite the claim that this paper illustrates “progressive visualization” this aspect was not a strong focus except in the sense that tasks were given in phases. A striking, relevant result, resonating with that of Gellert and Steinbring with slightly older children, is the strong influence of feedback from peers. The children in Margolinas and Wozniak’s study had to represent their necklaces to peers so that they could reconstruct them, and the feedback from peers in this activity was influential in modifying the *frames* the children drew upon (to use the construct of Steenpass and Steinbring—Margolinas and Wozniak did not use this terminology).

The paper by Teppo and van den Heuvel-Panhuizen (2014) concludes the section on use of number lines as an epistemological tool. This carefully analyzed account provides excellent reference material encapsulating “a family of number line models, based on visual aspects of number lines each reflecting different forms and functions.” They describe clearly the different forms (visual and representational) and functions (operational) of the following: the filled number line, the empty number line, the directed-length number line, the rational number line, and the proportional number line. This paper is essential foundational reading for anyone wanting to investigate ways of using the visual tool of number lines in teaching concepts involving number.

Although Rivera’s paper (Rivera 2014) also deals with the learning of number operations by children, it is much more overtly about processes of visualization than are the four papers that precede it. The description in the abstract concerns “a longitudinal account ... from the initial visual processing phase to the converted final phase in numeric form”—words that could imply a movement away from visual forms of cognition to more *developed* mathematical forms, reminiscent of Bruner’s (1964) representational

development hypothesis, from enactive to iconic to symbolic thinking, in which visualization gets left behind. However, Rivera does acknowledge that visualization may continue to be useful (strongly emphasized also in my research results); but he downplays the role of visual thinking to providing “empirical support when appropriate” in the later stages. He does recognize the strength of the interplay between visual and nonvisual modes (again, also in my research results), using abductive, inductive, and deductive reasoning in conjunction with visualization. I could resonate well with his use of Duval’s (2002) theoretical ideas to describe inter-register conversions and intra-register treatments: The *kinds* of visualization within a visual register may become less context bound, thereby increasing their usefulness as learning develops. In my research there was a more abstract form of visualization, which I called *pattern imagery*, which was more effective in learning mathematics than was the *concrete imagery* that tied associations to one case or picture.

In the context of interactive technology use, Abrahamson et al. (2014) provide an intriguing account of the use of dynamic visualization to illustrate the polysemy inherent in situations of visual representation. The twelfth of my thirteen questions (introduction) deals with the dynamic context made possible by such technology. The computer game-like setting of their tasks enables “the emergence of proportional-equivalence notions” as learners confront the rule that one of their hands is raised upwards at double the rate of the other. It would be interesting to know what idiosyncratic imagery was experienced by these learners, resulting in their “noticing” (Mason 2002) of different aspects of the polysemic visual display. However, such a result was beyond the scope of the methodology employed.

The paper by David et al. (2014) illustrates nicely how a teacher’s use of a visual display on the board, accompanied by the metaphor of a “shower”, in illustrating the use of the distributive law in early algebra, can become over-generalized and used incorrectly by students. This teacher became aware of the students’ misuse, and worked hard to correct it. However, the authors point out that there could be cases where a teacher is not aware of potential problems of interpretation on the part of students. A visual display is *memorable*! Herein lies potential strength, but also the possibility of over-generalization; thus awareness that visualization is not self-explanatory is very important for teachers.

Working in the Palestinian educational system, Natsheh and Karsenty (2014) introduce relationships between a function and its derivative by means of the visual display of two graphs (a function and its derivative, without saying which is which), which students have to interpret, thereby deepening their awareness of certain properties. The rich variety of interpretations by the students—on which aspects they focus their attention—again shows the polysemy of such visual displays. The work of these authors also illustrates the

value of tasks such as those they gave to promote reflection in connecting different registers (Duval 1999) in learning a mathematical topic (question 5 in my list of 13 questions).

The final two papers, both in the context of special education, contribute broader perspectives to the epistemological questions addressed in this issue. Figueiras and Arcavi (2014) examine the haptic perspectives of two blind students and one partially sighted student learning about solids of revolution with a skilled blind teacher. I was already aware, having visited a school for blind students in England, that far from being absent, visual imagery is a very important component of their mathematics learning. But a surprising and fascinating revelation from the research of these authors is that the usual relationship between source (vehicle) and target (tenor) domains in use of metaphors in mathematics, is *inverted* for these blind students. Instead of using a familiar context from the world around them as a source of metaphor to help in understanding a mathematical notion, these students are able to use their understanding of mathematical notions (e.g., the shape of a solid) as a source of metaphor to cast light on what something in the world *looks* like! The tenor and the vehicle of the metaphor are reversed in such a case (question 6 in my list of 13 questions). I would have liked some discussion of how these authors conceptualize sense of *touch* as “visual” (as in the title of this issue of *ZDM*). Nevertheless it is apparent that there *are* connections between these two sense modalities; more analysis of these connections would be useful.

Finally, the paper by van Garderen et al. (2014) examines the efficacy of self-generated drawings by learners in grades 4–7 and contrasts the “challenges” confronted by students designated as being with or without learning disabilities. These authors identify nine “challenges related to the use of diagram proficiency” in solving mathematics problems given in verbal form, grouped under the subheadings of conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. This work seems to relate to my ninth question of the 13, namely how such self-generated drawings may facilitate or hinder the reification of processes as mathematical objects. These authors are also aware that “imagery considered schematic” (pattern imagery) may in many cases be more effective in mathematics learning than “imagery considered pictorial” (concrete imagery).

The following section addresses aspects that these papers have in common.

### 3 What converging knowledge about visualization in mathematics thinking and learning do we gain from the papers?

A strong general impression that emerges from reading all of these papers is that it is inadequate for a teacher to

present a visual display in teaching some mathematical topic, and expect learners to make the connections with the mathematical principle that is in the teacher’s mind. The need for teacher *awareness* of the polysemy of visual artifacts is a theme that comes through repeatedly. These issues implicate several of the questions in my list (questions 1, 2, 5, 6, and 9), but particularly question 7, *How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions?* This question concerns Duval’s (1999, 2002) concern with making connections within a particular mathematical register (the intra-register treatments), but also the need for inter-register conversions amongst different registers (question 5 in my list). The need is clear, and this need is a point of convergence in these papers: but how to promote such connections in mathematics classrooms at various levels is less specific as converging knowledge.

Several of the papers—especially those by Steenpass and Steinbring, Gellert and Steinbring, Margolinas and Wozniak, David et al., and Figueiras and Arcavi—suggest that a particular sociocultural climate in the classroom, namely one that encourages sharing of perspectives among learners, and with the teacher, is conducive to awareness of alternative perspectives. This issue implicates my second question, *What aspects of classroom cultures promote the active use of effective visual thinking in mathematics?* But this awareness is only the beginning. How to reconcile alternative interpretations and bring them flexibly into the canonical mathematics knowledge domain remains an issue only hints of which are given in some of the papers, e.g., the attempt by the teacher in David et al.’s account to resolve the students’ over-generalized use of the “shower” metaphor in the distributive law.

There is one question in my list, question 9—*How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects?*—that is hinted at in the results reported in many of the papers: in particular, in all of the four papers concerning use of variants of the number line, and David et al., and van Garderen et al. But a much finer grain of research focused on the visual properties *as such* is needed to address this question (more is said about this issue in the final section).

### 4 What (else) is missing? What should we be concerned about after reading the papers? What other questions are pertinent to ask?

As suggested in the previous section, a different focus, one that emphasizes the properties of visualization as a process and a product in fine-grained research, is necessary to move

beyond the general results reported in some of the papers. Some of the aspects that did not seem to be addressed deeply concern the following questions:

3. What aspects of the use of different types of imagery and visualization are effective in mathematical problem solving at various levels?
4. What are the roles of gestures in mathematical visualization?
8. How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions, and conventional mathematical processes and notations?
10. How may visualization be harnessed to promote mathematical abstraction and generalization?
11. How may the affect generated by personal imagery be harnessed by teachers to increase the enjoyment of learning and doing mathematics?
12. How do visual aspects of computer technology change the dynamics of the learning of mathematics?
13. What is the structure and what are the components of an overarching theory of visualization for mathematics education?

Question 4, the roles of gestures, is an important one that is partially addressed in the papers by Abrahamson et al., David et al., and Figueiras and Arcavi. Gestures were an important indicator of the presence of visual imagery of teachers and students in my original research (Presmeg 1985). Taking into account the growing research on the topic of gestures and embodied cognition in recent decades (e.g., Radford 2009) would require a special issue with this focus in its own right. With regard to question 8, idiosyncratic *interpretations* are described in many of the papers in this issue; however, only Rivera's paper explicitly addresses the issue of how students' self-generated drawings (reflecting idiosyncratic imagery) may be implicated in moving through a visual phase to canonical numeric forms. The issue of visual mental imagery is a difficult one to investigate, because researchers cannot be certain that they are accurately interpreting what is in another person's mind (see the final section), but in general, a finer grain of research on the properties of individual visual cognition is needed to inform how teachers may overcome the difficulties and harness the strengths of visualization in mathematics classrooms.

The issues of compartmentalization and prototypes (Presmeg 1992), abstraction and generalization (Presmeg 1997; question 10), affective aspects of visual cognition (question 11), and overarching theory of visualization (question 13) receive scant attention in these papers. The dynamic possibilities of interactive computer technology (question 12) receive attention only in the paper by Abrahamson et al. This field is another in which recent research would require a special issue in its own right. However,

despite what is *not* addressed in the papers in this issue, the focus on the complexity of issues raised when teachers use visualization as an epistemological tool in helping students to learn mathematics is an important and significant contribution.

### 5 More importantly, how can we move the field forward in both aspects of theory and methodology?

In endeavoring to reach a finer grain in research on visualization as an epistemological learning tool, it is useful to take into account the different properties of visual and verbal symbolism. Already in the 1970s, psychological research identified many of these distinctions (e.g., Paivio 1971; Pylyshyn 1973; Kaufmann 1979; Kosslyn 1980). The properties of visual thinking are particularly fruitful in *creativity* (e.g., Einstein 1970, 1979). Skemp (1971, p. 111) made an early summary, which is a useful starting point, in attending to the "contrasting and largely complementary properties" of visual and verbal symbolism:

Visual symbolism	Verbal symbolism
Abstracts specific properties, e.g., shape, position	Abstracts properties independent of spatial configuration, e.g., number
Harder to communicate	Easier to communicate
More individual	More socialized
Integrative, showing structure	Analytic, showing detail
Simultaneous	Sequential
Intuitive	Logical

Many of the distinctions involved in these properties are implicit in the research reported in the papers of this issue of *ZDM*: it would be useful to have them spelled out and investigated in more detail. For instance, the papers that report results of research on the number line identify the *positional* properties of number forms, suggesting the reflexive relationship between visual and verbal symbolism: a step beyond Skemp's classification.

Engaging with the interesting reports of research into various uses of visualization as an epistemological learning tool in mathematics in this issue, has caused me to realize that a further set of questions related in a finer grain to the *teacher's role* in such learning, could be generated. Just for a start, here are a few suggestions:

*For prospective teachers at all levels to become aware of the affordances and challenges of using visualization as a learning tool in mathematics education, what aspects of teacher education programs are effective?*

*If teachers become aware of individual differences in ways that learners use visual cognition (e.g., through individual task-based interviews), could such awareness be used to enhance the effectiveness of their use of visual tools in teaching mathematics?*

*How may teachers guide learners to use more effective pattern-based visualization, rather than visualization that is context-bound, in their mathematical problem solving?*

One paper in this issue (Figueiras and Arcavi) also suggests that there is much more to learn about the power and pitfalls of using the haptic system in teaching mathematics. How are touch and sight related in various relevant contexts? Despite the growing literature on gestures and embodied cognition, conceptual frameworks for research that links visual and haptic modalities are still in need of development.

Another area that is growing intensely is research on ways that uses of interactive technology change the learning of mathematics. In many cases, this research has not focused specifically on the connections with visualization as an epistemological tool: more studies in this area could help us to understand the power and pitfalls of dynamic visualization in this regard. The paper by Abrahamson et al. is a useful start.

My final question of the thirteen questions concerned the structure and components of an overarching theory of visualization in mathematics education. Given that theory is a tool for structuring research, and given the range of possible research questions still in need of investigation, I am not sure that *one* overarching theory could meet the need. However, to obviate the fragmentation caused by the absence of such a theory—illustrated in the diversity of conceptual frameworks used in the research reported in this issue—it would be productive at least to explore definitions of visualization for the purpose of establishing some theoretical common ground. Such an exploration was started in the collaboration reported by Nardi (2014). There is still much to be learned from the methodology and careful analysis of early researchers such as Krutetskii (1976) in the Soviet Union. He not only classified a large collection of mathematical tasks into various categories related to visualization, but used these tasks in individual interviews with students of a range of preferences and abilities. The fine grain of his work enabled him to generate theory that could still be useful today. His methodology is also informative: think-aloud procedures in individual interviews enabled him to obtain a fairly deep understanding of individual differences in visual processing of the students he interviewed. It is apparent that researchers can never be completely confident that they have ‘captured’ the thought processes of another human being—and the very process of asking might have changed the thinking—but the usefulness of such methodology is evident in the carefully

recorded insights he reported, and in the theoretical edifice that he built (Nardi 2014).

For the sake of coherence in research, and to carry the field forward, it is necessary to take into account what has already been accomplished in a particular field, such as *visualization as an epistemological learning tool in mathematics*, and hence to address newer questions that inevitably emerge, starting with—but by no means confined to—those I have suggested.

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