RESEARCH PROPOSAL for PhD

RHODES UNIVERSITY EDUCATION DEPARTMENT (Mathematics Education)

Candidate: Deepak Pravin Mavani Student Number: 615P0005

Supervisor: Prof. Marc Schäfer

Provisional Title

A critical analysis of whether visualisation capabilities in dynamic geometric software develop meaning making in terms of conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners.

Abstract

Visualisation plays a central role in developing mathematical ideas because it can be used to make mathematical ideas explicit and thus has the potential to advance understanding. This study centres around the GeoGebra Literacy Initiative Project (GLIP) a teacher development project in Mthatha. The aim of this project is to grow and develop appropriate ICT skills in participating teachers to harness the teaching and learning potential of GeoGebra. GeoGebra is a freely available software package that is very interactive and makes use of powerful features to create images that are dynamic and can be moved around the computer screen for mathematical exploration. This research project is located within GLIP and analyses whether GeoGebra applets develop conceptual and procedural understanding in selected Grade 11 learners. One aspect of GLIP is for teachers to use GeoGebra applets that they have developed themselves and implement them in their classrooms in pre-determined cycles that are aligned to the curriculum. My study will focus on the learners and explore whether learning has taken place in terms of developing mathematical meaning making. This interpretive research study is designed as a case-study of selected learners' engagement with these GeoGebra applets in learning mathematical concepts. The case is a cohort of selected Grade 11 learners who have been taught by GLIP teachers, and my unit of analysis is the learners' interaction with the applets. My data will consist mainly of video recorded observations and interviews. The learners' engagement with the applets will be analysed against two strands of Kilpatrick et al's mathematical proficiency, namely conceptual understanding and procedural knowledge. The theoretical orientation of this study is constructivist learning.

Keywords: Dynamic Geometry Software, Visualisation, Mathematical Proficiency, Meaning Making, Constructivism.

Common Statement

This PhD study, with a focus on **learning**, is paired with another PhD study, also within the GLIP, that focuses on aspects of **teaching** with GeoGebra applets in the context of using visualisation in mathematics education.

PROVISIONAL TITLE

A critical analysis of whether visualisation capabilities of dynamic geometry software develop meaning making in terms of conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners.

1 Introduction & Background

Mathematics is one of the key learning areas in schools across the globe. It helps to develop mental processes that enhance logical and critical thinking, and problem solving skills that can contribute to decision-making. Mathematical problem solving enables us to understand the world around us and teaches us to think creatively (Dept. of Basic Education, 2011). The partnership between ICT and Mathematics has recently opened up exciting teaching, learning and research opportunities. ICT[1] has spread its wings across the globe and is accessible to many individuals. The South African census of 2011 (Statistics SA, 2012) shows that 24.1% of households in South Africa own computers and the Research ICT Africa (RIA) survey 2011-12 (Gillwald, Moyo, & Stork, 2012) shows that 44,4% of the population have access to computers in South Africa. The entry of digital technologies into some classrooms, thus has become inevitable. The Africa Institute of South Africa (AISA), under the HSRC[2], a South African statutory research agency, proposed in 2012 that the Department of Basic Education (DBE) and Department of Higher Education and Training (DHET) in South Africa adopt appropriate measures to use ICT as a means of enhancing education in the country (Policy Brief # 80, Aug 2012). The National Strategy for implementing Mathematics, Science and Technology (MST) acknowledged the significance of ICT in education as a teaching and learning resource, but the ministerial committee report of 2013 found that there is 'poor evidence of educational technology' in many schools (Dept of Basic Education, 2013, p. 8). Technology equipment such as computers and tablets provided are under-utilised because many teachers are not sufficiently equipped for effective use of this technology in the classroom (Dept of Basic Education, 2013). There are many educational software resources available on the market, but many of them are expensive and require annual license fees. This research projects is firmly rooted in the free open source software community and actively supports the use of software like GeoGebra that offers an 'excellent classroom-teaching tool' (Dept of Basic Education, 2013) for Mathematics. Kaput (1992) asserted that digital technologies afford new ways of learning mathematics.

1.1 GLIP

GeoGebra is one of the dynamic geometric software technological resources that is available to mathematics teachers to be used in mathematics classrooms. The GeoGebra Literacy Initiative Project (GLIP) is a teacher development project in Mthatha in the Eastern Cape that aims to equip teachers and students with the necessary skills to use technological tools, particularly GeoGebra, for teaching and learning of mathematics. The project was launched in November 2015 with the initial participation of 12 mathematics teachers from a secondary school in Mthatha, Eastern Cape. The first phase of GLIP consisted of introductory training in GeoGebra and

[1] Information and Communication Technology
[2] Human Sciences Research Council

helping the participating teachers to familiarise themselves with the software. During this phase, teachers were trained to work and explore with GeoGebra and to think about how they could use GeoGebra to realise its potential in the mathematics classroom. The participating teachers were workshopped to develop skills in GeoGebra for about 12 hours spread across a 2week period. This training enabled the teachers to generate their own GeoGebra applets. These are short programs (I will explain applets in detail under Technology 2.2 below) that can be used for classroom demonstration and student exploration. The workshop materials were adapted from Hohenwarter's (2009) and permission has been sought to use them for training purposes.

During the second week of March 2016, this team of trained teachers in turn trained their learners in the basics of GeoGebra. In the second phase of GLIP which will begin in August 2016, the participating teachers will start using GeoGebra applets that they have collaboratively developed in their mathematics classrooms. It is envisaged that they will create approximately 8 applets that are directly aligned with their teaching plan and the curriculum in the course of 2016/2017. My research focus will be on the learners as they engage with these applets, whereas my fellow researcher will focus on the teachers implementing these applets. Both studies will foreground the harnessing of the visualisation opportunities to enhance conceptual understanding of these applets firstly in the context of teaching (my fellow researcher's study) and secondly in the context of learning (the focus of my study)

2 Literature Review

2.1 Conceptual landscape – Visualisation

Clements (1982) wrote that many highly original and significant creations of the human mind have been largely the result of nonverbal mental representations. Mathematical ability requires generalised and abstract thought (Krutetskii, 1976) and to establish a 'mathematical advantage', one should perceive and use clear mental pictures. A visual-pictorial approach to solve problems is a powerful tool for pupils to use when engaging with mathematical ideas and reasoning. Krutetskii (1976) quotes Kolmogorov "Wherever possible, mathematicians strive to make the problems they are studying visual geometrically." In my experience in the classroom, for those who prefer to think visually, even the most complicated problems can be done quickly and accurately in the mind. Krutetskii (1976) suggests that it is productive to capitalise on the visual aspect of an abstract mathematical idea. What is visualisation?

Giaquinto (2007, p. 35) claims that concepts are often triggered by experiences of seeing or visual imagining. Reflection on visual experiences may lead to new mathematical beliefs and ideas. For Duval (2014, p. 160) visualisation "is the recognition, more or less spontaneous and quick, of what is mathematically relevant in any visual representation given or produced."

Tall (1994) suggests that visualisation is a powerful process to convey a large amount of data holistically and enables an individual to change focus from one part of the picture to another so that relationships may be observed visually and pictorially. While Guzman's (2002) version of mathematical visualisation is, "…a way of acting with explicit attention to the possible concrete representations of the objects one is manipulating in order to have a more efficient approach to the abstract relationships one is handling" Zimmermann & Cunningham (1991) defined visualisation as mental images formed mentally, with pencil and paper, or with the aid of technology.

Arcavi (2003) attempts to provide a comprehensive definition of visualisation and reflects upon the rich role it can play in the teaching and learning of mathematics. He proposes that:

> "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings".

The chief characteristic of the above definition that is germane to learning, is the emphasis on 'ability' and 'process", which resonates with Bishop (1988) who claims that visualisation is both a product (phenomena) and a process (the activity). For the purpose of this study, Arcavi's (2003) definition is central. For my work it effectively synthesis ideas of many researchers (Bishop, 1980; Duval, 1999; Zimmermann & Cunningham, 1991) and applies to the central theme of this study.

2.1.1 Classification of visualisers and non-visualisers

Lean & Clements (1981) classified mathematics students into three broad groups namely 'visualisers', 'verbalisers' and 'mixers'. Visualisers are individuals who habitually employ visual imagery to solve problems. Verbalisers are those who tend to use verbal codes in problem solving. Mixers do not have a preference for the verbal or visual mode. Presmeg (1986) in her study also classified learners (and teachers) into visualisers and non-visualisers. As with the study of Lean & Clements (1981) Presmeg found that the majority of her participants that performed well in Mathematics had a tendency to use non-visual methods in their problem solving strategies. Presmeg identified the following possible factors to explain her findings: 1) The non-visualisers are able to generalise ideas rapidly and thus curtail their procedures in solving problems. They do not feel the need for concrete images. 2) School mathematics curriculum favours the nonvisual thinkers as achievement is measured in time bound tests. Visualisers tend to use time consuming procedures, and hence struggle to perform well in tests. 3) The teaching emphasis is usually on nonvisual representations and very often visual representations are dispensed with whenever possible.

The results of the study by Krutetskii (1976) proposed that the ability to visualise is not a necessary component to be mathematically able. He suggests that the "analaytic type" of students prefers to use verbal-logical methods and have no need for visual support. The "geometric type" prefers to use visual-pictorial methods even when the problem could easliy be solved by reasoning. For the "harmonic type" of student there apperas to be a relative equilibrium of well developed verbal-logic and visual-pictorial processes evident when solving mathematical problems. Kruteski (1976) warns that there needs to be a balance between emphasising verbal-logical and visual-pictorial schemes as the dominance of one can, for example hinder the generalisation process. From the above literature, it is evident that there exist individual differences in learners in how they use visualisation.

2.2 Relavance of Visualisation

In contrast to the above literature, Arcavi (2003) and Zimmermann & Cunningham (1991) have foregrounded the importance of visualisation and suggested that in mathematics visualisation enables us to ‘see the unseen’. This is significant in mathematics as many mathematical concepts are abstract and, in my experience rather ‘invisible’ to young learners. Duval (1999) asserts that visualisation in mathematics is indeed needed because it can display organization of relations and enables learners to form images of complex mathematical ideas. When learners encounter complicated problems that involve complex procedures, visual representations can simulate a whole solution (Arcavi, 2003; Krutetskii, 1976). Hilbert (1952) asserted that “With the aid of visual imagination, we can illuminate the manifold facts and problems of geometry.”

The ANA[3] diagnostic report of 2014 stated that learners across all provinces in South Africa found topics on Geometry difficult to answer (Dept of Basic Education, 2014, pg.53). Bishop (1988) quotes Hoz’ (1981) work and identifies what he calls "geometrical rigidity" caused by a child being unable to 'see' a diagram in a different way. Fischbein (1993) clarified that a geometrical figure is not a mere concept. All geometric figures possess conceptual properties (like abstractness, generality) and figural properties – like shape, magnitude. Fischbein (1993) defines figural concepts as “mental entities” in the investigation and manipulation of objects. I concur with Fischbein (1993) that visualisation is a fusion of conceptual and figural properties of geometric shapes. Consider the following diagram in Figure 1. *In a circle with centre O, the diameters are drawn perpendicular to each other. P is any point on the circle and PQ and PS are drawn perpendicular to the diameter. What is the length of QS?*

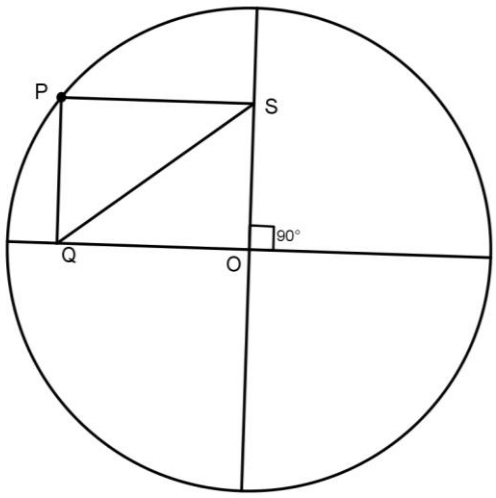


Figure 1

[3] Annual National Assessment

At a first glance, it appears that the data is insufficient to solve this problem. The radius is not given, the position of P is not known, and hence the lengths of PQ and PS are unknown. The moment we visualise that PQOS is a rectangle however, we can ‘see’ that PO = QS because they are diagonals of a rectangle, and hence conclude that QS = OP = radius. Fischbein (1993) then declares that ‘the fusion between concept and figure is complete’. Here visualisation becomes very useful to recognise the configuration of a rectangle and thus the relationships of the other lines in the problem.

While logical reasoning is a step-by-step procedure, visualisation is the grasp of the whole (Duval, 2013). For Duval (2013), visualisation refers to a cognitive activity and he proposes three kinds of cognitive activities in geometry: firstly, seeing and recognising shapes, secondly measuring, calculating and comparing magnitude, and thirdly, inferring from properties. This is illustrated in Figure 2 below.

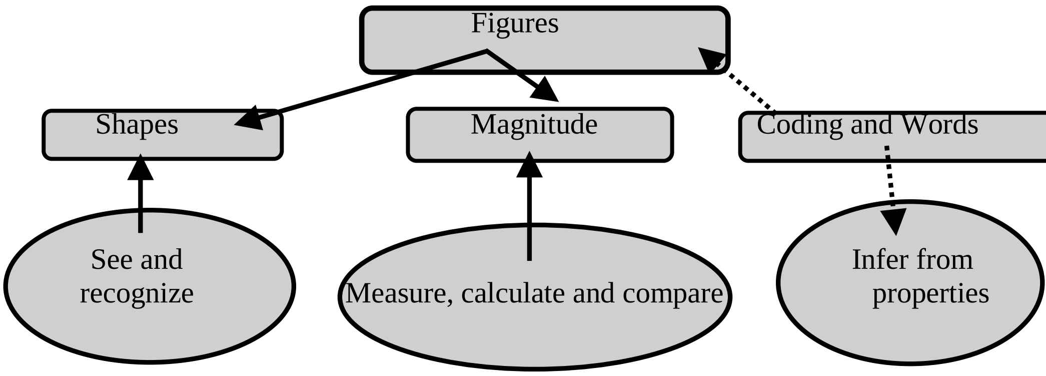


Figure 2 The three faces of figure in geometry

Seeing is recognizing a shape and object which Duval (2013) identifies as ‘perceptual recognition’. Duval considers perceptual recognition as an ‘out of mathematics’ visual representation as the given properties in the object are not considered. Many students have difficulty in understanding geometric figures by themselves because of the perceptual recognition of shapes and figures is so predominant and spontaneous that it impedes mathematical ways of visualising the figures. Many students, for example, fail to recognise a square when it is tilted. The process of visualisation can help students to realise that a square is not only an image but is a shape controlled by its definition (or properties).

Mathematical visualisation lies in the implicit selection of which visual units within the configuration are relevant and which are not. Visualisation requires one to move from the whole to some of the visual units that are characteristic features of the represented phenomena (Duval, 1999). Geometrical properties correspond to what remains invariant when the drawing is changed, by moving either one of its points or its segments. Visualization in geometry is specifically related to the cognitive activity of seeing shapes within a figure. In order to achieve this, the notion of *figural unit* is introduced which enables an individual to decompose and recompose a given figure. Duval (2013) defines *figural units* as:

> The figural units are all the elements, which can be visually discriminated in any constructed figure. These elements are not characterized by their shape but by the number of their dimension.

He further argues that a “mathematical property cannot be visualised by a single figural unit, but only by the visual relation between two or more figural units.” We can promote processes of visualisation among learners, by making them identify the various line segments, points of intersection in any geometrical figure. Therefore, in any geometrical activity visualisation and the corresponding problem solving discourse ought to be linked, technical words are to be matched with figural units. “The mathematical recognition of objects represented in figures depends on the given properties and not only on the perceptual recognition of shapes.” Conceptual development in geometrical problems require both visual and conceptual understanding (Mhlolo & Schäfer, 2013).

2.3 Technology – dynamic geometric software

When combined with visualisation processes, ICT technologies in my experience can bring about an exciting paradigm shift in mathematics education which allows for powerful multiple representations of mathematical concepts as visual objects. The National Curriculum Statement (2011) aims “to produce learners that are able to use science and technology effectively and communicate effectively using the visual and symbolic in various modes.” Noss & Hoyles (1996) find in their work that “effective learning of conceptually-based material involving the appropriation of mathematical relationships occurred where there was a synergy of interdependence and autonomy through active construction at the computer” (p.150). Other researchers (Bhagat & Chang, 2015; Mariotti, 2000) also concluded that the use of technology can significantly improve learning.

Arcavi (2003) emphasised the potential role of technology in visualisation. The dynamic features of technology provide an opportunity for learners to see what is not necessarily obvious to the everyday eye. For example, an image of a graph enables learners to ‘see’ the relationships between quantities. It may also sharpen our understanding and serve as a springboard for questions which we were not able to formulate before. I will discuss later how technology aided learning promotes making conjectures.

Ruthven, Hennessy, & Deaney (2008) argue that dynamic geometry enables students to work with figures *easier, faster and more accurately*. An important feature of dynamic geometry software (DGS) is that it can be designed around mathematical principles and inspire students to interact with it. When the emphasis of the lesson is on promoting students’ broad understanding, then DGS can develop awareness of mathematical ideas through exploring dynamic figures. In my experience DGS can be a powerful and effective tool to guide students to discover mathematical properties by themselves if used appropriately. The tools of DGS like dragging, which is discussed later, enable learners to move shapes around the computer screen and thus discover their inherent characteristics. There are many DGS tools available, like Cabri Geometry, Geometer’s Sketchpad, GeoGebra. For the purpose of this study, I will be employing the DGS, *GeoGebra*[4]. “GeoGebra is a multiplatform mathematics software package that gives everyone the chance to experience the extraordinary insights that maths makes possible” (“GeoGebra,” n.d.). GeoGebra is an open-source dynamic

[4] Marucs Hohenwarter the founder of GeoGebra refers to it as a dynamic mathematics software.

geometry software package that combines algebra and geometry. With GeoGebra, abstract ideas may be reified not only by making connections between the real and abstract world, but also between different mathematical concepts. The software runs virtually on any operating system as it requires only a Java plug-in and, unlike commercial products, students and teachers are not constrained by licenses to run the software on only a limited number of computers (Hohenwarter & Hohenwarter, 2008). As observed earlier, the Department of Education also recommended the use of GeoGebra as a technological tool for teaching and learning of mathematics.

2.3.1 Virtual Manipulatives and Applets

In mathematics, a manipulative is an object or artefact which is designed so that the learner can explore and interact with some mathematical concept by manipulating it. Just as a picture can be worth a thousand words, manipulatives can provide visual representations of ideas, helping students to learn and to understand mathematics. In my experience as a mathematics teacher, encouraging learners to play with manipulatives can stimulate multisensory experiences such as touch and vision, provide immediate access to ideas and concepts, and offer multiple entry points for discussions and reasoning.

When considering virtual manipulatives Moyer (2002) writes that they offer learners powerful opportunities to manipulate a virtual object on the computer screen. Virtual manipulatives can be used to make meaning of mathematical concepts and see relationships between mathematical properties as a result of one's own action and input. Virtual manipulatives can be very interactive. They can visually represent an object that can be dynamically manipulated and be used for constructing shapes. This deep engagement with an object, in my view can enhance conceptual understanding of that object.

In DGS these dynamic artefacts allow the learners to engage and control the inherent actions of the objects in DGS by pointing, clicking and dragging aspects of the shape on the computer screen. The strength of DGS visualisations is that the movement of the representation preserves the invariant relationships in the visual representation when dragging and moving elements of the representation. This allows for making conjectures and verify hypotheses (Kaput, 1992). An integral component of GeoGebra is the use of applets. Merriam-Webster ("Merriam-Webster," n.d.) define an applet as "a small program designed to be executed from within the application." Applets are usually short and user-friendly applications. Applets can be used to provide interactive features enabling the user to change the dimension and the characteristics of the entire graphic or parts thereof. Thus it is very suitable for demonstration, visualisation and teaching ("Java Platform," n.d.). GeoGebra applets are representations of mathematical ideas which anyone can interact with. Often, these are small programs made to serve a specific purpose. For example, Figure 3 and Figure 4 are screen shots of an applet designed specifically for learners to explore the relationship between the x-intercepts and the turning points of the graphs of cubic functions (Mavani & Mavani, 2014).

In figure 3, the intercepts of a cubic function are at points A, B and C and turning points are D_1 and D_2. This applet allows the learners to drag the intercepts along the appropriate axes. They are encouraged to make conjectures, for example, about what would happen to the turning points when the intercepts approach each other. They can explore and test their conjectures dynamically and discover that, as in Figure 4, that the point of coincidence of two intercepts must necessarily be a turning point. Thus, GeoGebra applets are virtual manipulatives providing an opportunity to explore a variety of mathematical concepts through hands-on manipulation of functions, graphs, formulas, shapes and figures. This applet can also be accessed online at http://ggbm.at/XXMem7fU.

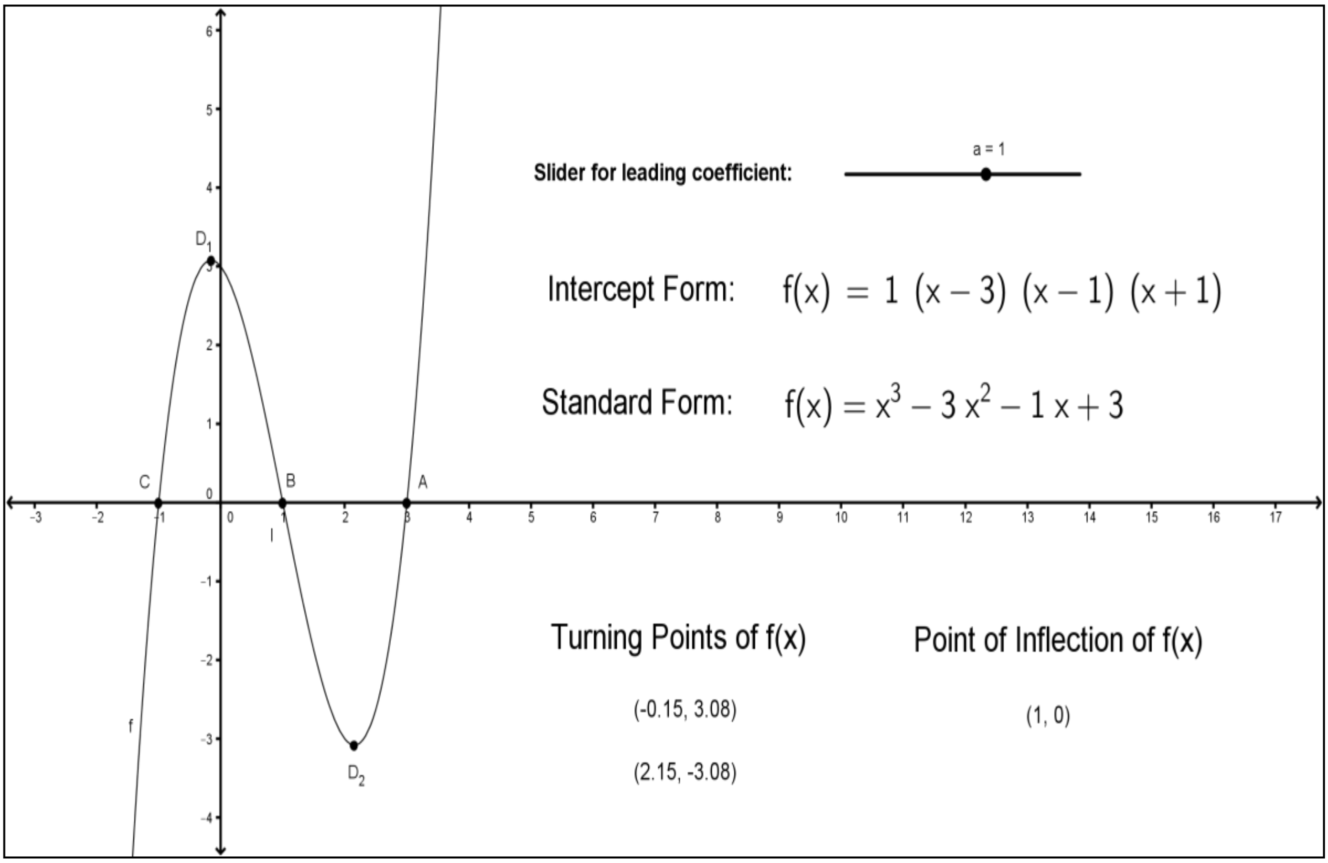


Figure 3

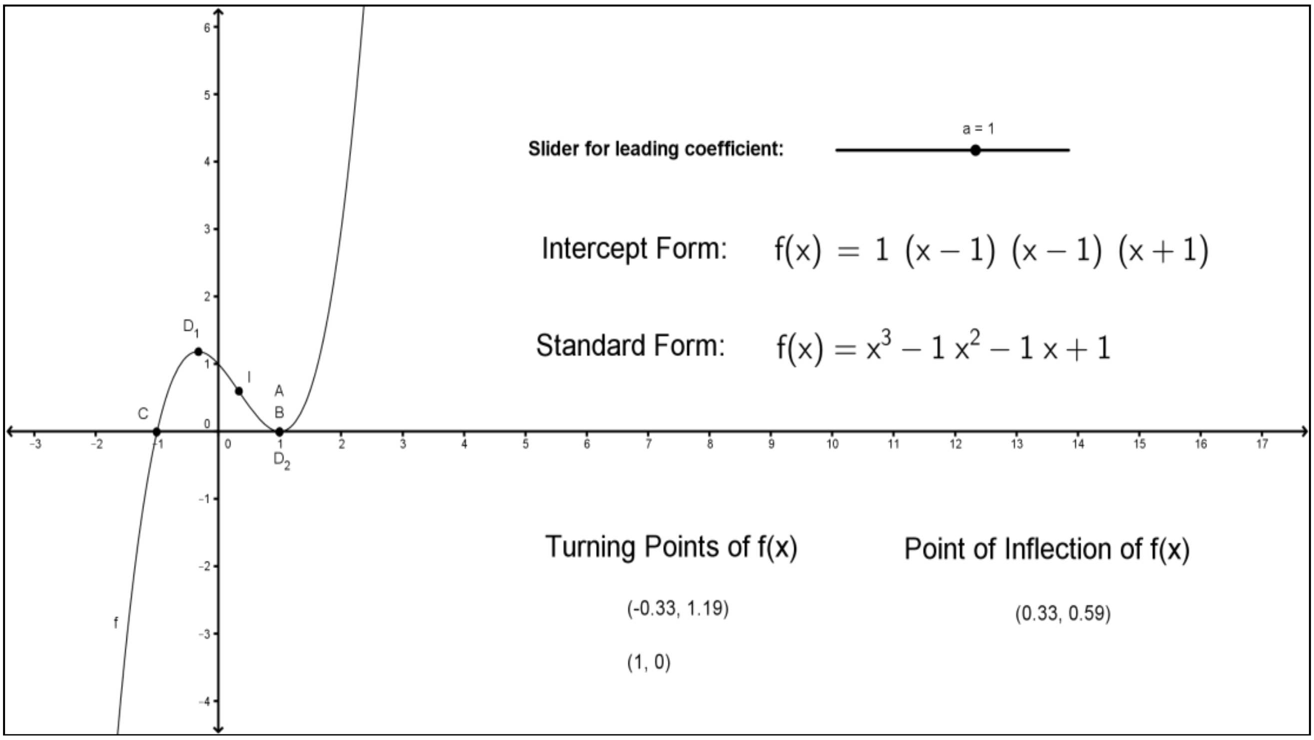


Figure 4

According to Duffin (2010) and Gage (2010) the strengths of using applets is that firstly, they can complement a discourse with visual demonstration which would have been difficult to convey in a traditional manner. Secondly, these short applications enable dynamic visualisation of mathematical concepts, allowing the learner to explore relationships. GeoGebra applets provide the visual aspect that is missing from text alone, thereby helping visual learners. Thirdly, representations can be linked to help draw attention to the relationships between two representations and thus deepen understanding. Lastly, the technical advantage is that these can be easily embedded in a web-based application, hence these can be accessed by anyone from anywhere. GeoGebra applets can facilitate experimentation and hence create an interactive environment of "learning by doing".

2.4 Mathematical Proficiency

2.4.1 Conceptual Knowledge and Procedural Knowledge

Much has been researched and written about the conceptual and procedural knowledge dichotomy (Hiebert & Lefevre, 1986; Van De Walle, 2004). Skemp (1989) expounds that in mathematical learning 'relational understanding' would lead to better transfer and wider applications of the procedural knowledge. Van De Walle (2004) succinctly comments that 'understanding is a measure of the quality of connections that a new idea has with existing ideas'.

However Silver (1986) argues that these two types of knowledge are inextricably linked. Procedural knowledge must be based on conceptual understanding and procedural fluency is necessary for developing conceptual knowledge. There is a constant interplay between conceptual and procedural knowledge when engaged in tasks that require some level of understanding. His research analysis led him to conclude that the concepts of conceptual and procedural knowledge are intertwined.

2.4.2 Strands of mathematical proficiency

Developing a more comprehensive framework for mathematical learning and proficiency Kilpatrick, Swafford, & Findell (2001) concur that students become more proficient when they understand the underlying concepts of mathematics and when they are skilled at computational procedures. Kilpatrick et.al (2001) identified a set of five intertwined strands, called strands of 'mathematical proficiency' that are important in developing this proficiency. These strands are not independent, but interwoven, representing different aspects of the whole (pg. 16). Due to the scope of this study, only two strands of mathematical proficiency are considered viz. **conceptual understanding** and **procedural fluency**.

According to Kilpatrick et al. (2001) **conceptual understanding** is the comprehension of mathematical concepts, operations and relations. It refers to the grasp of fundamental mathematical ideas, much more than the isolated facts and procedures. When mathematical ideas are represented mentally in an organised manner (connected and structured), retrieval of knowledge is easy. Thus learning with understanding is more powerful than simply memorizing because the organization improves retention, promotes fluency, and facilitates learning related material (pg.118). **Procedural fluency** refers to the skills in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately. It refers not only to the execution of procedures competently, but also to awareness of various computational tools, and selection of the appropriate tools in a given situation. Conceptual understanding and procedural fluency are interwoven. Understanding makes learning skills easier, while procedures help strengthen and develop that mathematical understanding. The knowledge of procedures with understanding provides a foundation for generating new knowledge. Thus both strands are critical to developing mathematical proficiency.

Mathematical meaning making is inherent in developing mathematical proficiency (Carter et al., 2009, p. 12). Making meaning and conceptual understanding are closely and tightly interrelated. Carter et al. (2009, p. 13) argue that in the absence of reasoning and sense making, "students carry out procedures correctly but may also capriciously invoke incorrect or baseless rules." As an example, I find grade 11 students when calculating the 'mean' of two numbers say, 21 and 32, often enter "21+32÷2" into their calculators and obtaining 37, not realising that a mistake has been made. Here the students did not appreciate the hierarchy of operations nor did they realise that the average of two numbers should lie midway between the numbers. According to Carter et al. (2009, p. 13), mathematical proficiency can only be achieved "by engaging in mathematical reasoning and meaning making as they learn mathematical content." By using technology such as GeoGebra as an integral part of mathematics classroom, they observe that "technology allows multiple representations to be linked dynamically, it can provide new opportunities for students to take mathematically meaningful actions."

The scholarly works of Arcavi (2003), Presmeg (1986, 2014), Duval (1999, 2013) and other researchers have provided compelling empirical evidence of the important role of visualisation in developing understanding of mathematical concepts and knowledge. Their research found strong evidence that learners do employ visual strategies (though different) to construct meaningful conceptual ideas. In particular, Presmeg (1985) commending the use of visual imagery, claims that the embodiment of abstract ideas in a concrete image can be effective in learning mathematics, ie the use of visualisation can facilitate learning of mathematical content with understanding.

Howson (2005) observes that a key aspect of mathematics teaching is to provide meaning to mathematical ideas through real life contextualisation and develop students' mathematical thinking. When students are exposed to different activities, different meanings area scribed to mathematical objects. Therefore, as a teacher, a variety of contexts need to be planned so that students' construction of meaning does not become deficient. The teacher may eventually move towards a more abstract view. Howson (2005) argues that as educators we are not only interested in mathematical 'end products' but also in the educational journey and the insights gained en route.

I concur with Haapasalo (2007) and argue in this study that technology-based learning provides a link between conceptual and procedural knowledge. The tools inherent in GeoGebra such as sliding and dragging, allow the learner to manipulate mathematical objects, illustrating different forms of mathematical representation. With the aid of multiple perspectives, like Algebra, Geometry, Calculus etc., in GeoGebra, multiple forms of representation can be visualised and connected, thus these two types of knowledge develop iteratively. By utilizing the dynamic visualisation power of technology, Haapasalo (2007) found with empirical evidence that "a mathematical concept can be built through simultaneous activation of conceptual and procedural knowledge."

2.5 Significance of the study

The goal of this research study is to investigate and explore whether GeoGebra visualisation applets develop conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners

There is little evidence in South Africa, especially in Eastern Cape, that GeoGebra is being used by mathematics teachers in their routine classes. The GeoGebra Institute at Pretoria (at University of Pretoria) and Port Elizabeth (at NMMU) are active with targeted workshops being conducted at regular intervals ("GeoGebra Institute Support Project (GIS)," n.d.). The website www.school-maths.com, maintained by the University of Pretoria contains more than 200 GeoGebra applets related to the CAPS curriculum. But in spite of the open-source programs available, there is little evidence in the literature that GeoGebra is used on a regular basis by South African learners. Also there is little evidence of its efficiency in teaching and learning of mathematics.

In the GLIP project, the specifically designed GeoGebra applets will be prepared by the participating teachers and used for teaching and learning to conceptually understand mathematical ideas and concepts. The development of these applets will by itself be a significant contribution to the teaching and learning community at large. In the process of developing and using these applets, and participating in this study many teachers and learners would have been empowered with technology-based teaching and learning and become familiar with dynamic software in particular.

My research goals and aims have been inspired by questions and gaps asked and identified by Duval (2014) and Presmeg (2014):

> The power of visualization of computers seems unlimited and has become pedagogically unavoidable. The question is not how to use them but what their impact is on the development of thinking and ability to use spontaneous mathematical knowledge for solving problems (Duval, 2014)

> How do visual aspects of computer technology change the dynamics of the learning of mathematics? …. the dynamic possibilities of interactive computer technology is another field in which recent research would require a special issue in its own right (Presmeg, 2014).

It is thus evident from the above, that there is a need to study learners' development of mathematical understanding and proficiency using visual aspects of dynamic geometry software.

2.6 Research Questions

The research questions that frame this study are:

- How are GeoGebra applets used as a learning tool to make mathematical meaning and develop understanding by selected Grade 11 learners?
- What visualisation role can GeoGebra play in the learning of Grade 11 mathematics?

3 Theoretical Perspective: Constructivism

This study is informed by a constructivist theoretical perspective. Based on Piaget's cognitive adaptation, von Glasersfeld (1990) summarised two principles of constructivism:

1) Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognising subject.

2) The function of cognition is adaptive, tending towards fit or viability.

In simpler terms, the first principle says that we all construct our own knowledge (Jaworski, 1994, p. 16; Van De Walle, 2004, p. 22). The learner's new knowledge draws on prior knowledge and experiences – learners are not blank slates. Constructivism is a theory of knowing that attempts to show that knowledge can only be generated by experience. This implies that learners construct their own knowledge by actively participating in a classroom situation. Arcavi (2003) sees visualisation as a process of constructing knowledge that promotes understanding. Therefore, visualisation 'as a process' provokes intellectual activity and construction of knowledge.

The second principle says that an individual learns by adapting. From a constructivist point of view, mathematical concepts are constructed that 'fit' or are viable with our real-world experiences. Giving credit to Piaget, von Glasersfeld (1981, 2000) argues that knowledge is constructed by an individual as an adaption to their subjective experience. Knowledge construction and adaptation are results of cognitive structuring, acknowledged by Piaget's genetic epistemology. Knowledge results from individual construction by modification of experience or idea. This principle emphasises that it is only possible to know the world through experiences.

However, Duval (2013) asserts that understanding involves grasping the whole structure, there is *no understanding without visualization*. While a single activity focusses on one or some units and properties of a mathematical concept, visualisation leads to "grasping directly the whole configuration of relation". For example, when an activity is confined to computation only, the learner may not look at the whole mathematical structure, but only on one aspect thereof. It is therefore important to provide learning opportunities for learners to make connections and experience the whole mathematical structure of a particular problem. As an example, Duval argues that the construction to draw graphs $y = 2x + 2$ and $y = x + 2$ by computing (say using a table) may not lead learners to understand the representation as a whole and the fact that the two graphs only differ from each other by virtue of their gradients. The use of visualisation enables learners to discriminate the equations conceptually. The applets designed in GLIP engage learners to explore and discover mathematical ideas and subtleties. Thus, technological tools like GeoGebra can provide visual learning experiences and allow learners to take control of their own learning (Bransford, Brown, & Cocking, 2000, p. 216). This aligns well with the principles of constructivism.

Hoyles (2005), in the context of technological learning, suggests that mathematical concepts need to be inextricably interwoven with applets. When planning a task for the learner that facilitates construction of mathematical ideas, careful attention is required in designing the applets to include the below two points. First, the tools embedded must be "just enough" to illuminate structures and relationships while not solving the task completely. Second, the task should be developed that foster students' engagement with mathematical ideas. Drawing on constructivist ideas, Hoyles observes that the student-computer interchange engage learner "in a dialectical relationship of action on the objects and thought." Thus the dynamic software tools can provide insight into mathematical structures and relationships. When learners are active participants in learning, using computer tools, they learn better and retain information longer. GeoGebra applets offer students an opportunity to do more than just listen to the teacher during class. Wright (2000, p. 139) argues that knowledge is the result of learners' activities rather than of the passive reception of instruction. Thus, from a constructivist perspective, activities designed in applets involve students in doing things as well as making them think what they are doing.

Recently research work of Jaworski et al. (2015) confirms that dynamic geometry software can engage students in deeper understanding of mathematical meanings. Her students provide mixed responses to GeoGebra, however. One group of students for example asserted that in-depth mathematical meaning is unnecessary as 'just because I understand maths better doesn't mean I'll pass the exam'. Another group of students reflected that using GeoGebra provided a dynamic visual representation helping them to 'spot patterns and trends' that would have been otherwise missed. In a technology-based constructivist classroom, there is less emphasis on transmitting information, but more on developing students' understanding of concepts and their skills. Promoting active learning promotes mastery of content with reasoning and understanding. In this study, the learners would be provided with selected applets created through GeoGebra and will be encouraged to engage with them to 'construct' their own meaning and verifying their own acquired knowledge.

The students' engagement in learning involves cognitive tasks and manipulation of conceptual ideas. Jonassen & Strobel (2006) argue that ICT can and should become the toolkit for meaning making and meaningful learning. Meaningful learning implies that learners are actively manipulating objects and tools, and during these activities they construct their own meanings and interpretations of their actions. Technology tools like GeoGebra can provide opportunities for learners to wrestle with objects, applets in this case. These opportunities not only develop understanding, but they also facilitate the adaptation of prior knowledge and experiences to new knowledge and meaning. When technological tools are used effectively, students can authenticate their understandings. Thus GeoGebra as used in this study, with its inherent exploratory possibilities, may provide opportunities for a heuristic approach to constructing mathematical meaning. The emerging technological tools, like DGS, allow students to engage in active learning processes in which they themselves build their own meanings and develop understanding (Khine, 2003). Hence technology enabled active learning aligns with the principles of constructivism as mentioned above.

4 Research Methodology

4.1 Orientation of the study

Merriam (2009) suggests that in qualitative research the emphasis is on "understanding how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences". This research is a study of learners' experiences when GeoGebra is used as a visualisation tool to teach and learn mathematical content. It focuses on learners' understandings as they construct mathematical meaning when engaging with GeoGebra applets. Their engagement with the mathematical content through their interactions with selected applets is analysed against the two strands of Kilpatrick et. al mathematical proficiency (2001) mentioned above.

Cohen, Manion & Morrison (2007, p. 21) assert that "to understand the subjective world of human experience" is fundamental in the context of an interpretative paradigm. Hence this research study involves making sense of learners' interaction with the dynamic software as they construct meaning of mathematical ideas. Further, Denzin & Lincoln (2005, p. 3) assert that qualitative research involves interpretive approaches in natural conditions, attempting to gain a better understanding of the participants and their actions, learners in this context.

Stake (2010, p. 63) observes that qualitative research is about how things happen and "happenings are experiences, and the researcher needs to probe the assertions until the experience is credible." Through observations and interactions with the selected learners, I hope to understand how they interpret or make conjectures of mathematical ideas when engaging with GeoGebra. In a given mathematical task, I need to probe and understand what meanings they make (Henning, 2005). The intention of this interpretative research study is to understand the mathematical learning experiences (Stake, 2010) of selected Grade 11 learners when engaged in tasks using technological tools.

4.1.1 Method – Case Study

Newby (2010, p. 51) considers a case study as a 'detailed analysis of an individual circumstance or event' that is chosen because something new is working. Contextually, as the DGS is introduced in selected classrooms, this study aims to make sense of 'possible processes' that learners may employ to understand mathematical objects. Newby suggests that researchers in a case study are also 'interested in variations from the expected'. This case study involves intensive, long-term interaction with selected learners, carefully recording the events in their classroom and "catch the unique features" (Cohen et al., 2007, p. 255), if any, of their interactions with GeoGebra applets. The case here is thus a cohort of grade 11 learners.

The units of analysis in a case study are usually the units of observation (Yin, 2009). Here the individual learner's interaction with GeoGebra is the unit of analysis. Yin (2009) states that the unit of analysis is where the researcher obtains the data for the case study. Stake (1995) believes that the temptation to be drawn away from the topic of study is one of the most serious problems in case study research. Hence the units of analysis need to be clearly determined so as to set boundaries for the research project.

Yin (2009, p. 34) defines a case study as a research process, “A case study is an empirical inquiry that investigates a phenomena in-depth and within its real-life context.” There is a particular need for my case study as the implementation of DGS is a relatively new phenomenon and requires rigorous and empirical reflection in order to understand the complexities of learning with this resource. The case study method allows me to retain the holistic and meaningful characteristics of real-life events (Yin, 2009).

4.2 Data collection in relation to GLIP

The school in which the GLIP project is operational will be my research location. The empirical field of the research is thus the GLIP programme. After the initial GeoGebra GLIP workshop with the participating teachers in November 2015, the feedback GLIP received was very encouraging and positive. The teachers wished to use GeoGebra as a teaching and learning resource in their mathematics classrooms. There was consensus among the teachers to develop and implement GeoGebra applets in Grade 10 and Grade 11 classrooms from early 2016. The teachers also requested school management to provide access to the computer lab for at least one hour per week for their mathematics classes. The training program for the learners has already started, and by the second week of August 2016 it will be complete. Soon after the learners’ training, the teachers will start using applets in the classrooms. This is the second phase of GLIP. It is in this phase of GLIP that my research project is located. Figure 5 below illustrates two of the cycles of the GLIP project. It also shows where my research project is located within each cycle. The empirical field of the research will be six of these GLIP cycles in Grade 11.

4.1 Sample Selection

My participants are the learners of the GLIP teachers who are implementing the applets that they have collaboratively developed. There are six grade 11 classes in the school, four of which who are taught by the GLIP teachers and two who are taught by me and my fellow researcher. I wish to select three participating learners from each of the four classes that are taught by the GLIP teachers. These 12 learners will be purposefully selected. Patton (2002, p. 47) argues that “logic and power of purposeful sampling derive from the emphasis on in-depth understanding.” I will select learners who are able to express themselves well and are able to articulate their thinking and understanding. They need not be ‘star performers’. These learners will be selected on the basis of their participation during their GLIP training and in consultation with their mathematics teachers. Sampling in my case is thus aimed at gaining rich insight about a specific phenomenon (learning with applets) and not for empirical generalisations from a sample to a population (Patton, 2002). I will obtain ethical clearances from the school management, teachers, children and their parents. The four teachers of these classes are the participants of my fellow researcher in her study.

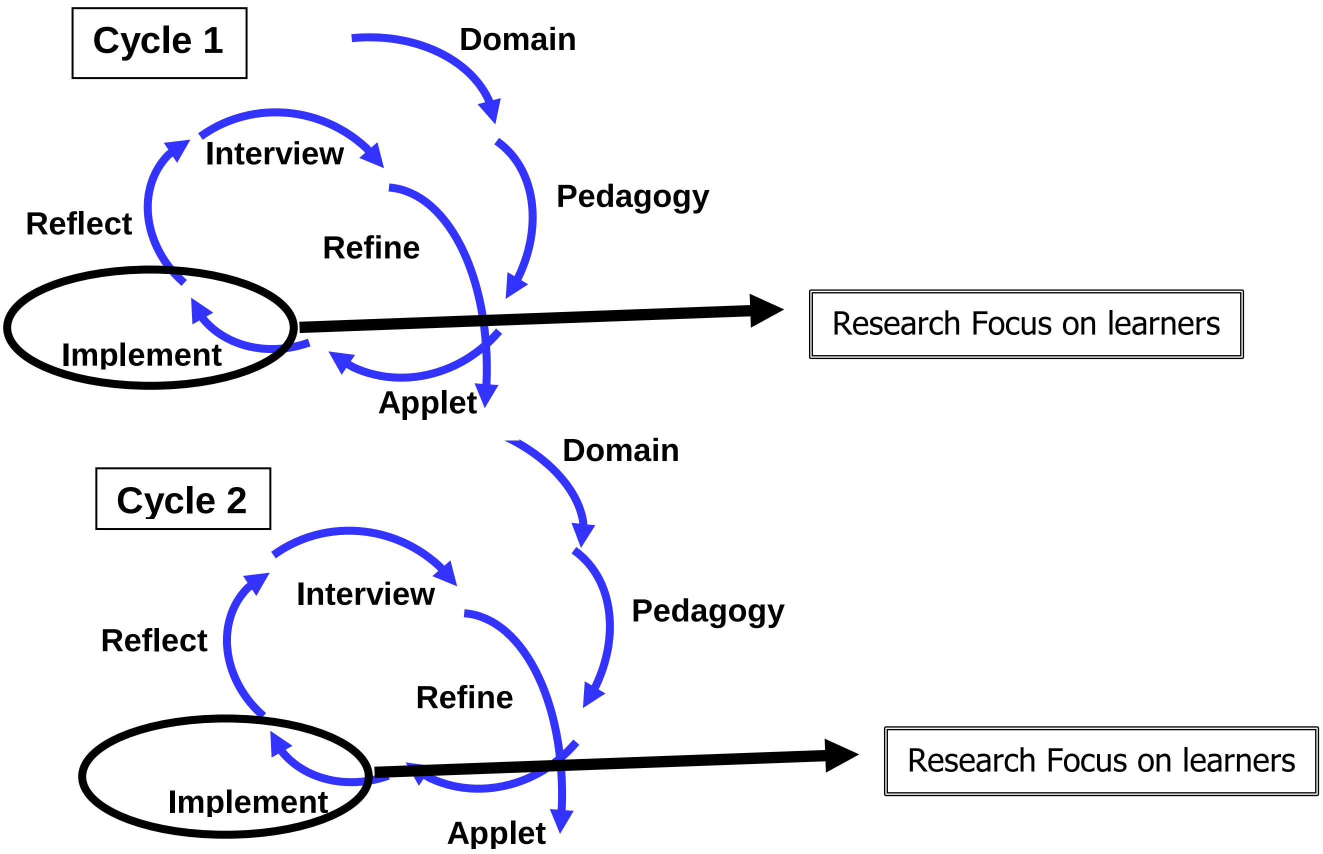


Figure 5: GLIP cycles in the empirical field of study

4.2 Research design

A preliminary interaction with the learners prior to data gathering is necessary to establish a rapport with them. During the training sessions of GLIP, my fellow researcher and myself will move around the classes offering help in GeoGebra. This would create familiarity and generate intimacy with all the classes. As the GLIP teachers implement their applets in their six cycles I will gather two sets of data from the 12 participating learners. Firstly, I will observe them as they interact with the applets and secondly I will interview each participant after each cycle. The first stage of data collection consists of two methods: a) Classroom Observation (Data A); and b) Screen Capture (Data B).

4.2.1 Classroom observation – Data A

In order to observe my participants, I will video record the learners interacting with the applets. My role will thus be one of a non-participant observer (Cohen et al., 2007). The specific focus of the video recordings will be to capture how the learners respond to the tasks that the teachers give them and their subsequent interaction with the applets. I specifically wish to record how they interact with the visualisations of each of the applets.

4.2.2 Screen Capture Videos – Data B

I also intend to capture the participants' workings with the computer using Cam Studio software (open-source) or any other screen capturing software. Screen capturing software enables the recording of the computer screen activities. The participants will be told about the screen capture of their working with the computer. I will hold an hour long awareness session with all the participants to explain the process of data collection with them. This will be held in the computer lab of the school after normal school hours. My pilot study will guide me on the technical logistics of the hardware and software required for this technique. The purpose here is to gain insight and interpret students' meanings as they *construct* mathematical ideas and make sense of concepts on the computer screen. Through this technique, in conjunction with the video recordings, I will collect data on how they use GeoGebra and the selected applets.

My observation time table will depend entirely on the GLIP cycles. All things being equal I anticipate to video record 2 lessons in each cycle i.e. 12 lessons in the course of 2016 and 2017. As the GLIP lessons are planned for an entire year or more this will ensure that different grade 11 topics are covered providing rich data across a variety of mathematical domains.

4.2.3 Learner interviews – Data C

This will be by my second stage of data collection after reviewing and analysing the video recorded lessons and the screen capture data. I will conduct personal interviews with each of the selected learners at the end of every cycle of the GLIP. I will prompt the learners to reflect on and recollect the strategies used for solving the tasks he/she attempted in the class. For this I may use the 'stimulated recall' technique by reviewing the individual videos with the particular learner in order to facilitate a reflective process (O'Brien, 1993). My specific focus in the interviews is to reflect on how the visualisation processes facilitated constructive engagement with the applets and whether these contributed to making sense of mathematical concepts.

During this interview, I will use questions such as listed in Table 1 below. The pilot study will enable me to refine these questions. The objective of the interview is to gather complementary information to Data A and Data B on learner's engagement with the dynamic software and their understanding of mathematical concepts. It will enable me to elicit learner perceptions and as well as reasons for their methods and actions while doing tasks. Since the individual interviews may take 45-60 minutes, they will be conducted after school hours after obtaining consent from their parents. These interviews with learners will be audio-video recorded and transcribed for later analysis.

Questions	**Objective**
When we say *function* what do we mean?	Meaning making; Conceptual Understanding
How did you interact with the applets during the class?	Visualisation Role of GeoGebra
Why did you do like this? Why did this work?	Meaning and Understanding
Can you guess what will be …?	Understanding, procedural knowledge; Visualisation
Are you able to relate this problem or solution strategy with any previous learning or any other topic?	Making Connections and Meaning; Conceptual
Can you do this in a different way?	Procedural Knowledge; Visualisation;

Table 1 Semi-Structured Questions

4.3 Data analysis

The basic principle of qualitative research is that data analysis is conducted simultaneously with data collection (Coffey & Atkinson, cited in Maxwell, 2008). By analysing parallel to the data collection will enable me to reflect on my methods and strategy. The audio-video recording of the lesson and the screen capture video will be reviewed a number of times. The classroom observation data (audio-video and field notes), and computer screen capture video data will be transcribed as soon as possible.

4.3.1 Analytical Instrument - VAMPA

For the purpose of analysing the video and interview data, I have developed the Visualisation Applets Meaning Making Mathematical Proficiency Analysis (VAMPA) framework illustrated in Appendix (**Table 3**), a Likert-type scale.

Based on literature on meaning-making habits propounded by Carter et al. (2009) and characteristics of mathematical proficiency proposed by Kilpatrick et al. (2001), I have identified 7 broad indicators, BI1 to BI7, of Meaning Making (MM), Conceptual Understanding (CU) and Procedural Fluency (PF). For each of the broad indicator, from my experiences and the literature on meaning making and mathematical proficiency, I have identified different observable or traceable learning activities. I will use the VAMPA framework, **Table 3**, to analyse the nature and extent of the use of GeoGebra for each learnering activity. The broad indicators BI1 to BI3 indicates the attainment of MM and CU; BI4 and BI5 indicates the attainment of MM and PF; and BI6 and BI7 indicates the attainment of all the three, MM, CU and PF. For example, within the broad indicator BI7, 'reflecting on solution – considering the reasonableness of a solution', I have identified the following learner responses and activities: 1) estimate the results of computation, 2) verify the final answer, and 3) carry out the procedure in a different method. In total I have identified 18 such observable activities and categorised them under 7 broad indicators, BI1 to BI7. The VAMPA will be the guiding instruments for analysing my data to answer my research questions. The VAMPA is derived from the theoretical framework of meaning making and mathematical proficiency and is at the heart of my case study analysis. Yin (2009) observes that the theoretical orientation in a case study helps to sustains the specific focus on the data.

The VAMPA as it stands is a work in progress and more categories or codes may still emerge from the data, as Cohen et al (2007) warns "the codes themselves derive from the data responsively rather than being created pre-ordinately." Thus these analytical instruments may be refined as the research progresses.

The process of data analysis will involve two interconnected levels.

Level 1: Analysis of Classroom Observation (Data A): In this level, classroom observation videos will be analysed. The videos and the transcription will be reviewed to describe and document the unique or common patterns of understanding. What are learners doing in the class when GeoGebra applets are provided for exploration? Do they start exploring straight away or do they wait for instructions from the teacher? The videos and field notes will be reviewed against the developed indicators in VAMPA, **Table 3**. The data will be categorised by linking learner's responses and activities with the broad indicators mentioned in VAMPA.

Analysis of Screen Capture Videos (Data B): The screen capture videos comprise of the computer interactions of the selected learners. These videos will give me an idea of how do they engage with GeoGebra. Is the system idle for long? Do they access any other applications like paint, internet? The screen capture videos will also be analysed against the developed instruments discussed above. The analysis of screen capture videos will be used to uncover what happened during the lesson and interpret the learner's activities borne out of their mathematical thinking. This analysis leg will allow me to gain in-depth understanding of their actions and motives.

Level 2: Analysis of Reflective Interview (Data C): This will be the final leg of the data analysis process. The reflective interviews with the learners will help to clarify my initial analysis. This level sharpens my level 1 analysis of the two data sets, data A and B. Very importantly, those actions and ideas that are used by the learners to explore their assumptions and meanings will be recorded and extracted for evidence. The analysis of articulation of learners thinking processes will illuminate their mathematical thinking and meaning making.

The Table 2 below is a summary of my analysis process

Data Collection	**Method**	**Analytical Instrument**	**Level of analysis**	**Quantity of data**
Classroom Observation	Video Audio Recording (Data A)	VAMPA	Level 1	12 nos. (2 per cycle X 6 cycle)
Screen Capture	Video Recording (Data B)	VAMPA	Level 1	12 nos. (2 per cycle X 6 cycles)
Reflective Interview	Semi-structure questions; Video Recording (Data C)	Clarify and sharpen above analysis;	Level 2	36 nos. (6 learners per cycle X 6 cycles)

Table 2 Snapshot of Research Design

4.4 Validity

Validity in qualitative research demonstrates that the explanation of a particular event, which is a piece of research, can actually be sustained by the data (Cohen et al., 2007, p. 135). Maxwell (2008, p. 243) identifies that '*bias'* and '*reactivity*' are two common threats to validity. However, these are threats to validity which can be never be erased completely but can only be attenuated by attention to validity. *Bias* refers to ways in which data collection or analysis are distorted by the researcher's theory, values, or preconceptions (Maxwell, 2008, p. 243). Cohen et al. (2007, p. 144) state that 'reactivity effect' is that respondents behave differently when subjected to scrutiny or observation. An 'intensive, long term involvement' with the learners, will be one of the strategies to deal with these validity threat. Acknowledging Becker and Geer (1957), Maxwell (2008) explains that the sustained presence of the researcher in the natural setting can help rule out spurious associations. This research is spread over an academic year, hence the above mentioned threats will be minimised. The long term interaction also enables me to collect 'rich' data which are detailed and varied enough to provide a full and revealing picture of the events.

In a stimulated recall technique situation, to increase validity, researchers must minimise the time delay between event and recall (Lyle, 2003). The reflective interview will thus immediately follow the classroom observation. In order to analyse learner's mathematical meaning making and visualisation process, I make use of multiple data sources of evidence like classroom observation, screen capture videos and reflective interview. The use of multiple sources of evidence in case studies allows an investigator to address issues under study, a process of data triangulation (Yin, 2009). Further I am collecting data under different environments, normal classrooms and individual interviews, which is another form of data triangulation. Stake (1995) agrees that data triangulation is an effort to see if what we are observing and reporting carries the same meaning when found under different circumstances.

The tasks that are provided to the learners during the class, aligning with the principles of constructivism, are designed after deliberations with the experienced mathematics teachers. Many of these tasks will be adapted from the literature, or that appeared in national examinations. Hence such tasks are valid and tested. The VAMPA framework that has been adapted from scholarly works of Kilpatrick et. al and Carter et al (NCTM) will be tested during the pilot study before the actual data collection. Similar indicators for mathematical proficiency have been used by the researchers like Stephanus (2013) and Stott (2014) which already have been validated. Although the instruments already have an inherent level of validity, they will be refined as a result of the pilot.

RU FACULTY OF EDUCATION: ETHICAL APPROVAL APPLICATION

IMPORTANT: The following form needs to be completed by the researcher and submitted with their research proposal to the Education Higher Degrees Committee. The details to which this form relates should also be evident in the text of the proposal.

GENERAL PARTICULARS

MEd (Half thesis)		**MEd (Full thesis)**		**PhD**		**Other: Please specify**	

TITLE OF RESEARCH: **A study of whether GeoGebra visualisation applets develop meaning making in terms of conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners**

DEPARTMENT/INSTITUTE: **Mathematics Education**

DATE: [Submission to EHDC]

RESEARCHER: **Deepak Pravin Mavani**

SUPERVISOR/S: **Prof Marc Schafer**

ETHICS

NB: You must read the Faculty of Education Ethics Guideline ***prior*** to completing this form. Please indicate below how your research supports the indicated ethical principle:

Respect and dignity

The first stage of the research design is to obtain official permission (Cohen et al., 2007, p. 55) from the principal and school governing body to conduct the case study. I will explain to the school management and the teachers the possible benefits accruing from the research (Cohen et al., 2007, p. 52), like, the integration of technology in mathematics classrooms. Confidentiality and anonymity will be assured and maintained throughout the research project. The audio-visual media, the field notes, screen capture videos the interview transcripts and / or any other data collected will kept in a safe and secure place. Further the soft copies of such data would be stored in a password-protected file.

Transparency and honesty

O' Leary (2004) emphasises that informed consent implies that the participants, learners in this case, are voluntarily involved, have a full understanding of the commitment, activities involved and fully aware that they are under no obligation to continue their involvement to the end of the project. In order to ensure that learners' consent is voluntary and informed, a signed consent from each participant and his/ her guardian (Cohen et al., 2007) will be obtained. Wherever a child is not providing the signed consent letter, his / her actions will not be recorded and will not be included in my analysis (Fine and Sandstrom (1988) cited in Cohen et.al 2007). However, only those learners with informed consent will be selected for screen-capture and reflective interview. The learners would also be provided with a meaningful explanation of the research goals (Cohen et al., 2007) and what we want to achieve. I will also discuss with them what benefits they will derive from participating in this project.

Accountability and responsibility

Being a responsible researcher, I will give credit to and acknowledge all sources of materials and keep records of all sources. Indicators of accountable research are when the research process is transparent and open (O'Leary, 2004) for any other researcher to trace the research methods, without compromising the confidentiality of the participants. During the process of this study there is an opportunity for teachers and learners to enrich themselves with the educational technology and GeoGebra in particular.

Integrity and academic professionalism

The research findings will be discussed based on empirical data obtained during the study. I will take care that such discussions are a true representation of learners' actions and words. The data will not be disclosed to anyone not related to the study without the explicit consent of the participants. Professionalism relates to treating all participants with respect, dignity and fairness (Cohen et al., 2007). Even though I am dealing with the students who have less power and status than me, I will not influence them. Instead, I will encourage them to provide me a direct reflection of their experiences during interactions. As already mentioned, I am an active participant in GLIP but I assume the role of an observer inside the classroom. This case-study research will be directed by the principles of ethics that promote integrity and professionalism and as Maxwell (2008) believes ethical concerns are involved in *every* aspect of research design.

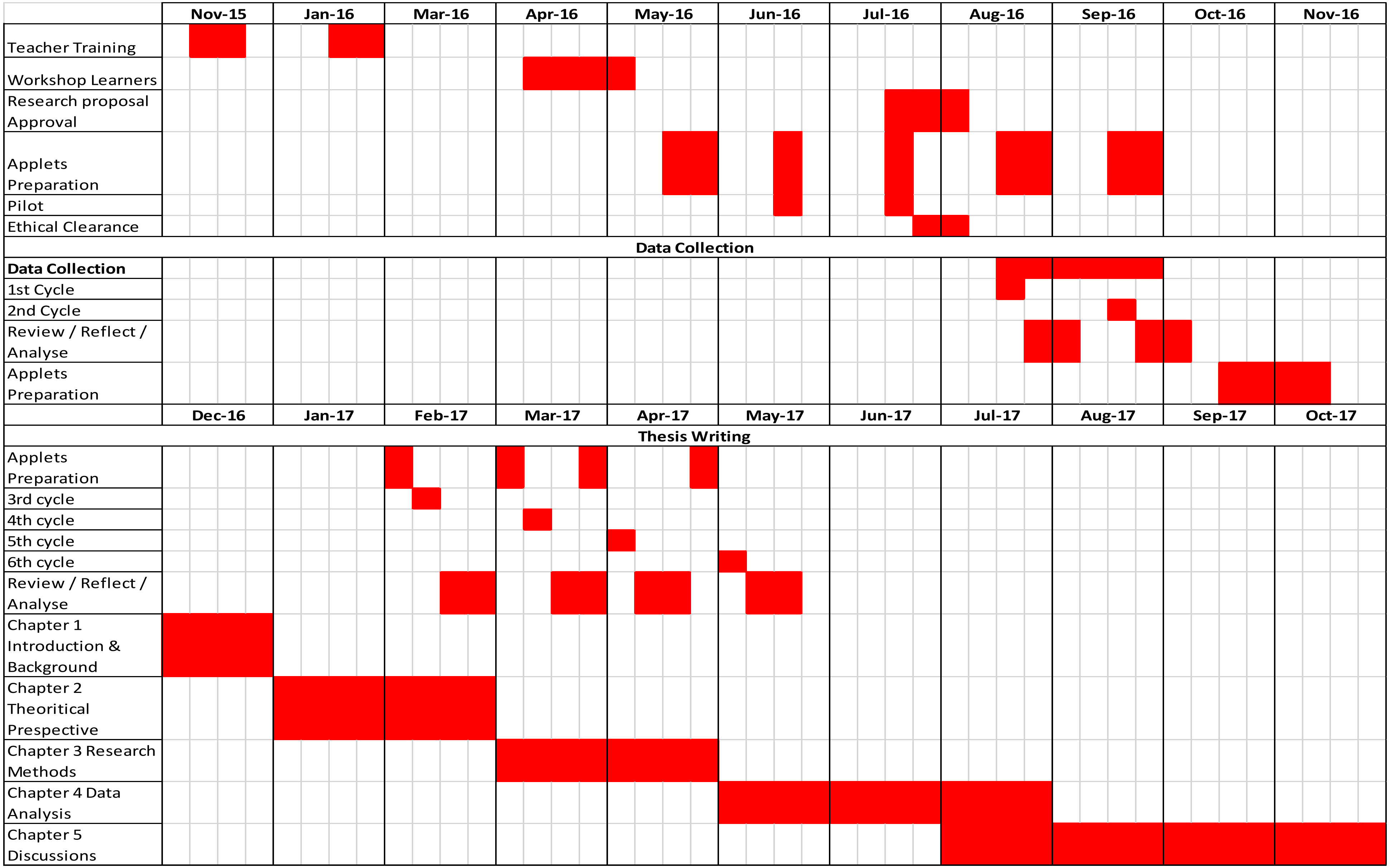
Signature
(Deepak Mavani)

Date: 27-06-2016

Signature
(Prof Marc Schäfer)

Place: Grahamstown

5 Proposed Research Schedule



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7 Appendix – Assessment Instruments

7.1.1 Visualisation-Applets- Meaning Making -Mathematical Proficiency Analysis (VAMPA) in Likert-type scale

Attained MM and / or MP	**Broad Indicators**	**Learners' responses and activities**	**Mode**	**Extent of use of GeoGebra**				
		Learner -		Never	Seldom	Sometimes	Very Often	Always
Meaning Making (MM) and Conceptual Understanding (CU)	**BI1:** Defining and identifying relevant mathematical concepts, relations and notations	Able to explain or demonstrate of concepts;						
		Able to represent mathematical ideas in multiple ways;						
		Provide examples;						
	BI2: Patterns and Relationships	Identifies relationships among concepts and make connections;						
		Understands variant and invariant properties;						
	BI3: Looking for Hidden Structure	Explores and discovers concepts;						
		Making and Verifying Conjectures based on explorations;						
		Able to think in curtailed manner;						
Meaning Making (MM) and	**BI4:** Purposeful use of procedures.	Shows appropriate, efficient and accurate execution of						

Procedural Fluency (PF)		mathematical procedures and algorithms;						
		Able to explain the procedures;						
		Able to select appropriate tool for a given problem;						
	BI5: Different approaches to solve a problem	Knowledge of multiple ways to estimate the solution;						
		Able to reverse a mental process (reverse train of thought) (Krutetskii, 1976, p. 88);						
Meaning Making (MM); Conceptual Understanding (CU) and Procedural Fluency (PF)	**BI6:** Applying Previously learnt concepts	Able to link to prior learning;						
		Employs basic procedures, formulae accurately and efficiently;						
	BI7: Reflecting on a solution – considering the reasonableness of a solution	Estimate the results of computation;						
		Verify the final answer;						
		Carries out the procedure in a different way;						

Table 3

Under column 'Mode' the visual or logical mode of approach will be noted as 'V'.

These activities are only indicative and may be refined after the pilot study and as the cycles progresses.